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An Introduction to the Theory of Local Twistors

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Abstract

This paper deals with the formalism of local twistors, which has developed from the twistor algebra, and extends some of the basic twistor concepts to curved space-time. Essentially, the central ideas are to define a twistor space at each point of the space-time, and to define a covariant derivative so that an operation of local twistor transport is possible; this leads to the definition of a conformally invariant curvature twistor. In an appendix, some conformally invariant spinors are discussed.

1. *Preliminary Discussion and Twistor Summary*

The formalism of local twistors has been developed as a method of applying the concepts of twistor algebra to curved space-time, since in the process of adapting these concepts to curved space-time certain difficulties become apparent; in fact, generalisation of the space of flat space-time twistors (global twistors), leaves it with only a weak symplectic structure, instead of the linear and complex analytic structure with which it is endowed in flat space-time; also, twistors which are non-null do not have a precise geometrical interpretation in curved space-time (Penrose, 1972a, b). These difficulties have led to consideration of other factors, still broadly based on underlying twistor motivations, resulting in the study of local twistors and asymptotic twistors. The local twistor theory, leading to the definition of a conformally invariant curvature twistor, is expounded in this article.

As stated above, in flat space-time twistor space possesses a linear and a complex analytic structure; the space of one-index twistors Z^α can be split up into the subspaces of twistors Z^α for which $Z^\alpha\bar{Z}_\alpha = 0$ (null twistors; $\bar{Z}_\alpha$ is the complex conjugate of Z^α), and those twistors Z^α for which $Z^\alpha\bar{Z}_\alpha \neq 0$. The null twistors can be represented as null straight lines in a suitably compactified Minkowski space-time, and the non-null twistors can be represented

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as null congruences (Robinson congruences) in this Minkowski space-time (Penrose, 1967, 1968a).

It can be shown that twistors in flat space-time can also be represented in spinor terms as solutions of the equation:

$$\nabla_{A'}^{(A}\omega^{B)} = 0 \tag{1.1}$$

which implies the existence of a constant spinor $\pi_{A'}$ such that:

$$\nabla_{A'}^{A}\omega^{B} = -i\epsilon_A{}^B\pi_{A'} \tag{1.2}$$

with

$$\nabla_{AA'}\pi_{B'} = 0 \tag{1.3}$$

A twistor Z^α in flat space-time is then represented as $Z^\alpha = (\omega^A, \pi_{A'})$ and in some spin-frame:

$$Z^0 = \omega^0, \qquad Z^1 = \omega^1, \qquad Z^2 = \pi_{0'}, \qquad Z^3 = \pi_{1'}$$

In order to make explicit the relationship between twistors and spinors, and to keep track of the indices in an expression containing both twistors and spinors, projection and injection operators (spinstors) have been introduced (Qadir, 1971); a twistor Z^α may then be written:

$$Z^\alpha = e^\alpha_A\,\omega^A + e^{\alpha A'}\pi_{A'} \tag{1.4}$$

with complex conjugate:

$$\bar{Z}_\alpha = e_{\alpha A'}\bar{\omega}^{A'} + e^A_\alpha\,\bar{\pi}_A \tag{1.5}$$

A spinstor therefore is endowed with a twistor and a spinor index (primed or unprimed).† In terms of components with respect to a local basis:

$$e^{\boldsymbol{\alpha}}{}_{\mathbf{A}}: e^0{}_0 = e^1{}_1 = 1; \qquad e^{\boldsymbol{\alpha}\mathbf{A}'}: e^{20'} = e^{31'} = 1$$

all others vanish. The following relationships hold:

$$e^{\boldsymbol{\alpha}}_{\mathbf{A}}: e^0{}_0 = e^1{}_1 = 1; \qquad e^{\boldsymbol{\alpha}\mathbf{A}'}: e_{20'} = e_{31'} = 1$$

all other vanish. The following relationships hold:

$$\left.\begin{aligned} e^\alpha{}_A e^B{}_\alpha &= \epsilon_A{}^B = e^B{}_\alpha e^\alpha{}_A \\ e_{\alpha A'}e^{\alpha B'} &= \epsilon_{A'}{}^{B'} = e^{\alpha B'}e_{\alpha A'} \\ e^{\alpha A'}e^B{}_\alpha &= e^B{}_\alpha e^{\alpha A'} = 0 \end{aligned}\right\} \text{ contraction over twistor index } \alpha$$

$$e^\alpha{}_A e^A{}_\beta + e^{\alpha A'}e_{\beta A'} = \delta^\alpha{}_\beta \quad \text{contraction over spinor indices } A, A'$$

† The notation for indices used here is that lower case Roman indices denote tensors, upper case Roman indices (primed or unprimed) denote spinors, and Greek indices denote twistors; components in some basis are denoted by bold indices.

Under a conformal rescaling, i.e. $g_{ab} \to \hat{g}_{ab} = \Omega^2 g_{ab}$, where Ω is a smooth, positive scalar field on the space-time, it can be shown that the equation $\nabla_{A'}^{(A}\omega^{B)} = 0$ is invariant, i.e.

$$\nabla_{A'}^{(A}\omega^{B)} = 0 \Rightarrow \hat{\nabla}_{A'}^{(A}\hat{\omega}^{B)} = 0$$

where $\hat{\omega}^B = \omega^B$ (Penrose, 1972a), and the vector $\xi^b = \omega^B \bar{\omega}^{B'}$ is a null conformal Killing vector. The conformal invariance of the equation $\nabla_{AA'}\omega^B = -i\epsilon_A{}^B\pi_{A'}$ then implies that $\hat{\pi}_{A'} = \pi_{A'} + i\gamma_{BA'}\omega^B$ where $\gamma_a = \Omega^{-1}\nabla_a\Omega$, and it follows that $\hat{\nabla}_{AA'}\hat{\pi}_{B'} = i\hat{P}_{ADA'B'}\hat{\omega}^D$ where†

$$P_{ab} = \tfrac{1}{2}R_{ab} - \tfrac{1}{12}Rg_{ab}$$

with

$$\hat{P}_{ABA'B'} = P_{ABA'B'} + \nabla_{AA'}\gamma_{BB'} - \gamma_{AB'}\gamma_{BA'}$$

Hence under a conformal rescaling of a flat space-time (for which $P_{ab} = 0$), the covariant derivative of the spinor $\pi_{A'}$ picks up a curvature term, and the equation:

$$\nabla_{AA'}\pi_{B'} = iP_{ADA'B'}\omega^D \tag{1.6}$$

is conformally covariant; in flat space-time it clearly reduces to (1.3). The conformally covariant spinor equations (1.2) and (1.6) will subsequently be encountered in the definition of a local twistor covariant derivative.

2. *Local Twistors*

In order to surmount some of the difficulties involved in adapting the theory of twistors to curved space-time, it is proposed that a twistor space be defined at each point of the space-time—thus a theory of local twistors is generated. However, this procedure introduces the points of the space-time into local twistor theory in a fundamental way, and there seems to be no obvious method of subsequently eliminating this dependence, which is rather alien to the spirit of twistor theory because for the purposes of this theory the attitude is adopted that it is the twistors themselves which are the basic entities, and space-time points are evolved at a later stage in the development of the theory (Penrose, 1972a). The local twistor theory is built up by using the spinor approach, and a local twistor Z^α at a point x of a curved space-time is represented as a pair of spinors ω^A and $\pi_{A'}$ defined at x. The local twistor space at each point is a fibre of the local twistor bundle, the symmetry group of which is $SU(2, 2)$, and the typical fibre is the space of global (flat space-time) twistors; local twistor space therefore has a linear and a complex analytic structure defined on it.

† The convention for curvature adopted here is that:

$$[\nabla_c, \nabla_d]\xi_b = R^a{}_{bcd}\xi_a$$

and $R^d{}_{adb} = R_{ab}$; in spinors $R_{ABA'B'} = -2\Phi_{ABA'B'} + 6\Lambda\epsilon_{AB}\epsilon_{A'B'}$ and $P_{ABA'B'} = \Lambda\epsilon_{AB}\epsilon_{A'B'} - \Phi_{ABA'B'}$ (see Penrose, 1968b).

A local twistor Z^α may be expressed in terms of its spinor parts thus:

$$Z^\alpha = e^\alpha{}_A Z^A + e^{\alpha A'} Z_{A'} \tag{2.1}$$

where the operators $e^\alpha{}_A$, $e^{\alpha A'}$ are as defined above, and in general give a non-constant correspondence between the spinor parts and the local twistors. (In equation (2.1) $Z_{A'}$ is a spinor such that in flat space-time $\nabla_{RS'} Z^A = -i\epsilon_R{}^A Z_{S'}$).

Under a conformal rescaling, the local twistor Z^α is invariant:

$$Z^\alpha \rightarrow \hat{Z}^\alpha = Z^\alpha$$

whilst the spinor parts transform as follows: $Z^A \rightarrow \hat{Z}^A = Z^A$ and $Z_{A'} \rightarrow \hat{Z}_{A'} = Z_{A'} + i\gamma_{BA'} Z^B$.

Hence the spinstors $e^\alpha{}_A$, $e^{\alpha A'}$ undergo the following transformations:

$$e^\alpha{}_A \rightarrow \hat{e}^\alpha{}_A = e^\alpha{}_A - i\gamma_{AB'} e^{\alpha B'} \tag{2.2}$$

$$e^{\alpha A'} \rightarrow \hat{e}^{\alpha A'} = e^{\alpha A'} \tag{2.3}$$

The complex conjugate expressions are:

$$e^A{}_\alpha \rightarrow \hat{e}^A{}_\alpha = e^A{}_\alpha \tag{2.4}$$

$$e_{\alpha A'} \rightarrow \hat{e}_{\alpha A'} = e_{\alpha A'} + i\gamma_{BA'} e^B{}_\alpha \tag{2.5}$$

Using these results it immediately follows that:

$$\delta^\alpha{}_\beta = \delta^\alpha{}_\beta \tag{2.6}$$

3. *Covariant Derivative*

The local twistor covariant derivative is defined by the equation:

$$\nabla^\sigma{}_\rho = e^R{}_\rho e^{\sigma S'} \nabla_{RS'} \tag{3.1}$$

where $\nabla_{RS'}$ is the spinor covariant derivative. The derivative (3.1) is the most obvious one to choose, and has a number of desirable properties, but it does not span the space of two-index local twistors, since a general two-index local twistor would have four non-zero spinor parts; as will be seen, the definition (3.1) leads to consideration of a six-index conformally invariant curvature twistor with only three non-zero spinor parts. $\nabla^\sigma{}_\rho$ defines a mapping of local twistor spaces:

$$\nabla^\sigma{}_\rho : \mathscr{T}^{\alpha \cdots \gamma}_{\beta \cdots \delta} \rightarrow \mathscr{T}^{\alpha \cdots \gamma \sigma}_{\beta \cdots \delta \rho}$$

and satisfies linearity and the Leibniz rule. This covariant derivative also commutes appropriately with contraction, complex conjugation, and index substitution, and has the additional properties:

$$\overline{\nabla^\sigma{}_\rho} = \overline{\nabla}^\rho{}_\sigma = e^S{}_\sigma e^{\rho R'} \nabla_{SR'} = \nabla^\rho{}_\sigma$$

i.e.

$$\nabla^{\sigma}{}_{\rho} = \overline{\nabla}^{\sigma}{}_{\rho} \text{ (hermiticity)}, \qquad \text{and} \qquad \nabla^{\sigma}{}_{\sigma} = e^{R}{}_{\sigma}\, e^{\sigma S'} \nabla_{RS'} = 0$$

A further important requirement is that the local twistor theory should be compatible with the global twistor theory of flat space-time, and for this reason two more properties of the local twistor covariant derivative are specified, thus:

(1) if the space-time is conformally flat, the equation $\nabla^{\sigma}{}_{\rho} Z^{\alpha} = 0$ has four linearly independent solutions over the field of complex numbers $\mathbb{C}$, which correspond to global twistors, i.e. the covariant derivative is integrable;

(2) the covariant derivative of a local twistor is conformally invariant, i.e.

$$\nabla^{\sigma}{}_{\rho} Z^{\alpha} = \hat{\nabla}^{\sigma}{}_{\rho} \hat{Z}^{\alpha}$$

These conditions are both satisfied by selecting a derivative which acts on the spinstors as follows:

$$\nabla_{RS'} e^{\alpha}{}_{A} = -iP_{RAS'B'} e^{\alpha B'} \tag{3.2}$$

$$\nabla_{RS'} e^{\alpha A'} = i\epsilon_{S'}{}^{A'} e^{\alpha}{}_{R} \tag{3.3}$$

with the complex conjugate expressions:

$$\nabla_{RS'} e_{\alpha A'} = iP_{RBS'A'} e^{B}{}_{\alpha} \tag{3.4}$$

$$\nabla_{RS'} e^{A}{}_{\alpha} = -i\epsilon_{R}{}^{A} e_{\alpha S'} \tag{3.5}$$

these equations being conformally covariant; unfortunately, it is unclear as to whether this procedure leads to a unique definition of a covariant derivative with the specified properties, but there seem to be no other simple spinor expressions which would suffice (presumably spinor terms involving the Weyl spinor Ψ_{ABCD} and its derivatives could be involved).

To show that the covariant derivative of Z^{α} has the stipulated properties, it is necessary to calculate the spinor parts of the derivative, as follows:

$$\begin{aligned} \nabla^{\sigma}{}_{\rho} Z^{\alpha} &= e^{R}{}_{\rho}\, e^{\sigma S'} \nabla_{RS'} (e^{\alpha}{}_{A} Z^{A} + e^{\alpha A'} Z_{A'}) \\ &= e^{R}{}_{\rho}\, e^{\sigma S'} \{ e^{\alpha}{}_{A} (\nabla_{RS'} Z^{A} + i\epsilon_{R}{}^{A} Z_{S'}) \\ &\quad + e^{\alpha A'} (\nabla_{RS'} Z_{A'} - iP_{RBS'A'} Z^{B}) \} \end{aligned}$$

using equations (3.2)–(3.5).

Therefore, the spinor parts of the covariant derivative are just the conformally covariant spinor equations (1.2) and (1.6) discussed earlier, and the conformal invariance of $\nabla^{\sigma}{}_{\rho} Z^{\alpha}$ stems from the covariance of these equations and that of the equations (3.2)–(3.5).

The spinor equations (1.2) and (1.6) which form the spinor parts of the local twistor covariant derivative define the operation of local twistor transport, which in flat and conformally flat space-times can be shown to be related to the operation of conformal Killing transport (Geroch, 1969, 1970). More specifically, if $\xi^a = \omega^A \bar{\omega}^{A'}$ is a null conformal Killing vector, and the conformal Killing transport equations for ξ^a are written in spinor form, in conformally flat space-time they are the same spinor equations as those defining the local twistor transport of a local twistor $Z^\alpha = (\omega^A, \pi_{A'})$ where $\nabla_{RS'}\omega^A = -i\epsilon_R{}^A\pi_{S'}$. It is hoped to extend this idea in a later paper.

The integrability condition for $\nabla^\sigma{}_\rho$ follows from the fact that in conformally flat space-time, spinors $Z^A, Z_{A'}$ can be found such that:

$$\nabla_{RS'}Z^A = -i\,\epsilon_R{}^A Z_{S'} \qquad \text{and} \qquad \nabla_{RS'}Z_{A'} = i P_{RBS'A'}Z^B$$

so that the local twistor Z^α defined from Z^A and $Z_{A'}$ obeys the condition $\nabla^\sigma{}_\rho Z^\alpha = 0$. Also, by a suitable conformal transformation, $P_{RBS'A'} = 0$ when $\nabla_{RS'}Z_{A'} = 0$. Then

$$\underset{x}{Z}_{A'} = \underset{0}{Z}_{A'}$$

$$\underset{x}{Z}^A = \underset{0}{Z}^A - ix^{AA'}\underset{0}{Z}_{A'}$$

where x^a is the position vector of the point x with respect to the origin O. Further, considering the equations defining the derivative of a spinstor, and by a suitable conformal transformation, $\nabla_{RS'}e^\alpha{}_A = 0$. Hence, there exist solutions of the spinstor equations given by:

$$\underset{x}{e}^\alpha{}_A = \underset{0}{e}^\alpha{}_A$$

$$\underset{x}{e}^{\alpha A'} = \underset{0}{e}^{\alpha A'} + ix^{AA'}\underset{0}{e}^\alpha{}_A$$

It is then evident that $\underset{x}{Z}^\alpha = \underset{0}{Z}^\alpha$, i.e. Z^α defines a global twistor through O and the covariant derivative is integrable.

4. *Torsion and Curvature*

A consequence of the equations (3.2)-(3.5) is that there exists a torsion twistor, i.e. $[\nabla^\mu{}_\lambda, \nabla^\sigma{}_\rho]\phi \neq 0$, for a scalar function ϕ, since $\nabla^\sigma{}_\rho\phi = e^R{}_\rho\, e^{\sigma S'}\nabla_{RS'}\phi$, and the second derivative is:

$$\begin{aligned}\nabla^\mu{}_\lambda\nabla^\sigma{}_\rho\phi &= e^L{}_\lambda\, e^{\mu M'}\nabla_{LM'}(e^R{}_\rho\, e^{\sigma S'}\nabla_{RS'}\phi)\\ &= e^L{}_\lambda\, e^{\mu M'}(e^R{}_\rho\, e^{\sigma S'}\nabla_{LM'}\ {}_{RS'}\phi - ie_{\rho M'}e^{\sigma S'}\epsilon_L{}^R\nabla_{RS'}\phi\\ &\quad + i\, e^\sigma{}_L e^R{}_\rho\, \epsilon_{M'}{}^{S'}\nabla_{RS'}\phi)\end{aligned}$$

Then

$$[\nabla^\mu{}_\lambda, \nabla^\sigma{}_\rho]\phi = i(\delta^\sigma{}_\lambda\nabla^\mu{}_\rho - \delta^\mu{}_\rho\nabla^\sigma{}_\lambda)\phi = T^{\mu\sigma\alpha}_{\lambda\rho\beta}\nabla^\beta{}_\alpha\phi$$

where

$$T^{\mu\sigma\alpha}_{\lambda\rho\beta} = i(\delta_\lambda{}^\sigma\delta_\rho{}^\alpha\delta_\beta{}^\mu - \delta_\rho{}^\mu\delta_\lambda{}^\alpha\delta_\beta{}^\sigma) \tag{4.1}$$

is the torsion twistor and has the following properties:

(i) $$T^{\mu\sigma\alpha}_{\lambda\rho\beta} = T[^{\mu\sigma\alpha}_{\lambda\rho\beta}] \tag{4.2}$$

The brackets here are interpreted as follows:

$$W[^{\alpha\gamma}_{\beta\sigma}] = \tfrac{1}{2}(W^{\alpha\gamma}_{\beta\sigma} - W^{\gamma\alpha}_{\sigma\beta})$$

also $$V(^{\alpha\gamma}_{\beta\sigma}) = \tfrac{1}{2}(V^{\alpha\gamma}_{\beta\sigma} + V^{\gamma\alpha}_{\sigma\beta})$$

(ii) $$T^{\lambda\sigma\alpha}_{\lambda\rho\beta} = 0 \tag{4.3}$$

(iii) $$T^{\mu\lambda\alpha}_{\lambda\rho\beta} = i(4\delta^{\alpha}{}_{\rho}\delta^{\mu}{}_{\beta} - \delta^{\mu}{}_{\rho}\delta^{\alpha}{}_{\beta}) \tag{4.4}$$

(iv) $T^{\mu\sigma\alpha}_{\lambda\rho\beta}$ is hermitian, i.e.

$$T^{\mu\sigma\alpha}_{\lambda\rho\beta} = \bar{T}^{\mu\sigma\alpha}_{\lambda\rho\beta} \tag{4.5}$$

where

$$\bar{T}^{\mu\sigma\alpha}_{\lambda\rho\beta} = \overline{T^{\lambda\rho\beta}_{\mu\sigma\alpha}}$$

(v) $T^{\mu\sigma\alpha}_{\lambda\rho\beta}$ is conformally invariant, i.e.

$$T^{\mu\sigma\alpha}_{\lambda\rho\beta} = \hat{T}^{\mu\sigma\alpha}_{\lambda\rho\beta} \tag{4.6}$$

This follows from (2.6).

(vi) $T^{\mu\sigma\alpha}_{\lambda\rho\beta}$ is covariantly constant, i.e.

$$\nabla^{\nu}{}_{\gamma} T^{\mu\sigma\alpha}_{\lambda\rho\beta} = 0 \tag{4.7}$$

this being essentially due to the fact that $\nabla^{\nu}{}_{\gamma}\delta^{\alpha}{}_{\beta} = 0$, which is easily demonstrated.

The second derivative and commutator of derivatives of a local twistor Z^{α} can be calculated and expressed in terms of spinor parts. The commutator in fact becomes:

$$[\nabla^{\mu}{}_{\lambda}, \nabla^{\sigma}{}_{\rho}]Z^{\alpha} = C^{\mu\sigma\alpha}_{\lambda\rho\beta} Z^{\beta} + T^{\mu\sigma\gamma}_{\lambda\rho\beta}\nabla^{\beta}{}_{\gamma}Z^{\alpha} \tag{4.8}$$

where

$$\begin{aligned} C^{\mu\sigma\alpha}_{\lambda\rho\beta} &= e^{L}{}_{\lambda}\, e^{\mu M'} e^{R}{}_{\rho}\, e^{\sigma S'}\{e^{\alpha}{}_{A}\{[\nabla_{LM'}, \nabla_{RS'}]Z^{A} \\ &+ (\epsilon_{L}{}^{A} P_{RBS'M'} - \epsilon_{R}{}^{A} P_{LBM'S'})Z^{B}\} + e^{\alpha A'}\{[\nabla_{LM'}, \nabla_{RS'}]Z_{A'} \\ &+ P_{LRM'A'}Z_{S'} - P_{RLS'A'}Z_{M'} - iZ^{B}(\nabla_{LM'}P_{RBS'A'} - \\ &\times \nabla_{RS'}P_{LBM'A'})\}\} \end{aligned} \tag{4.9}$$

The spinor parts of this expression can be simplified by using the following lemmas:

Lemma I

$$[\nabla_{LM'}, \nabla_{RS'}]Z^{A} + (\epsilon_{L}{}^{A} P_{RBS'M'} - \epsilon_{R}{}^{A} P_{LBM'S'})Z^{B} = \epsilon_{M'S'}\Psi_{B}{}^{A}{}_{LR}Z^{B}$$

Proof:

$$[\nabla_{LM'}, \nabla_{RS'}]Z^A = \epsilon_{M'S'}\Box_{LR} Z^A + \epsilon_{LR}\Box_{M'S'}Z^A$$

where:

$$\Box_{LR} = \nabla_{Y'(L}\nabla^{Y'}_{R)}$$

$$\Box_{LR}Z^A = (\Psi_B{}^A{}_{LR} - 2\Lambda\epsilon_{B(L}\epsilon_{R)}{}^A)Z^B$$

$$\Box_{M'S'}Z^A = \Phi_B{}^A{}_{M'S'}Z^B \qquad \text{(Penrose, 1968b)}$$

The result then follows since the terms in Λ disappear due to the skew-symmetry of the metric spinor, and the terms in $\Phi_{ABA'B'}$ reduce to a spinor expression skew in three indices.

Lemma II

$$[\nabla_{LM'}, \nabla_{RS'}]Z_{A'} + P_{LRM'A'}Z_{S'} - P_{RLS'A'}Z_{M'} = \epsilon_{RL}\bar{\Psi}^{B'}{}_{A'M'S'}Z_{B'}$$

The proof is similar to that in Lemma I.

Lemma III

$$\nabla_{RS'}P_{LBM'A'} - \nabla_{LM'}P_{RBS'A'} = \epsilon_{S'M'}\nabla^X{}_{A'}\Psi_{XRLB} + \epsilon_{RL}\nabla^{X'}{}_B\bar{\Psi}_{X'M'S'A'}$$

Proof

The spinor Bianchi identity may be written:

$$\nabla^X{}_{A'}\Psi_{XRLB} = \nabla^{M'}{}_R\Phi_{LBM'A'} - 2\,\epsilon_{R(L}\nabla_{B)A'}\Lambda$$

Using the symmetry properties of the Weyl spinor, it follows that:

$$2\nabla^X{}_{A'}\Psi_{XRLB} = -\nabla^{M'}{}_R P_{LBM'A'} - \nabla^{S'}{}_L P_{RBS'A'}$$

and from the properties of the metric spinor:

$$2\,\epsilon_{S'M'}\nabla^X{}_{A'}\Psi_{XRLB} =$$
$$(\nabla_{RS'}P_{LBM'A'} - \nabla_{LM'}P_{RBS'A'}) - (\nabla_{RM'}P_{LBS'A'} - \nabla_{LS'}P_{RBM'A'})$$

The required result is obtained on taking the complex conjugate of this, and combining the two expressions.

Collecting these results together,

$$C^{\mu\sigma\alpha}_{\lambda\rho\beta}Z^\beta =$$
$$e^L{}_\lambda e^{\mu M'} e^R{}_\rho e^{\sigma S'}\{e^\alpha{}_A(\epsilon_{M'S'}\Psi_B{}^A{}_{LR}Z^B) + e^{\alpha A'}[\epsilon_{RL}\bar{\Psi}^{B'}{}_{A'M'S'}Z_{B'}$$
$$+ iZ^B(\epsilon_{M'S'}\nabla^X{}_{A'}\Psi_{XRLB} + \epsilon_{RL}\nabla^{X'}{}_B\bar{\Psi}_{X'M'S'A'})]\}$$

and

$$C^{\mu\sigma\alpha}_{\lambda\rho\beta} =$$
$$e^L{}_\lambda e^{\mu M'} e^R{}_\rho e^{\sigma S'}\{e^B{}_\beta[e^\alpha{}_A\epsilon_{M'S'}\Psi_B{}^A{}_{LR} + ie^{\alpha A'}(\epsilon_{RL}\nabla^{X'}{}_B\bar{\Psi}_{X'M'S'A'}$$
$$+ \epsilon_{S'M'}\nabla^X{}_{A'}\Psi_{XRLB})] + e_{\beta B'}e^{\alpha A'}\epsilon_{RL}\bar{\Psi}^{B'}{}_{A'M'S'}\} \qquad (4.10)$$

However, $C^{\mu\sigma\alpha}_{\lambda\rho\beta}$ is anti-hermitian, i.e. $C^{\mu\sigma\alpha}_{\lambda\rho\beta} = -\bar{C}^{\mu\sigma\alpha}_{\lambda\rho\beta}$, so that the curvature twistor is defined as follows:

$$K^{\mu\sigma\alpha}_{\lambda\rho\beta} = iC^{\mu\sigma\alpha}_{\lambda\rho\beta} \tag{4.11}$$

which is hermitian. Hence:

$$[\nabla^{\mu}{}_{\lambda}, \nabla^{\sigma}{}_{\rho}]Z^{\alpha} = -iK^{\mu\sigma\alpha}_{\lambda\rho\beta}Z^{\beta} + T^{\mu\sigma\gamma}_{\lambda\rho\nu}\nabla^{\nu}{}_{\gamma}Z^{\alpha} \tag{4.12}$$

with the conjugate expression:

$$[\nabla^{\mu}{}_{\lambda}, \nabla^{\sigma}{}_{\rho}]\bar{Z}_{\alpha} = iK^{\mu\sigma\beta}_{\lambda\rho\alpha}\bar{Z}_{\beta} + T^{\mu\sigma\gamma}_{\lambda\rho\nu}\nabla^{\nu}{}_{\gamma}\bar{Z}_{\alpha} \tag{4.13}$$

It can be seen that the spinor parts of the curvature twistor involve, apart from the metric spinor, only the Weyl spinor and its contracted first derivative, together with the complex conjugates of these. The main properties of the curvature twistor are summarised below:

(i) skew-symmetry:

$$K^{\mu\sigma\alpha}_{\lambda\rho\beta} = K\,[{}^{\mu\sigma}_{\lambda\rho}]^{\alpha}{}_{\beta}; \quad K\,[{}^{\mu\sigma\alpha}_{\lambda\rho\beta}] = 0 \tag{4.14}$$

(ii) contraction on any two indices gives zero:

$$K^{\nu\sigma\alpha}_{\lambda\rho\nu} = 0 \tag{4.15}$$

(iii) in conformally flat space-time $K^{\mu\sigma\alpha}_{\lambda\rho\beta} = 0$;
(iv) $K^{\mu\sigma\alpha}_{\lambda\rho\beta}$ is hermitian, by definition;
(v) $K^{\mu\sigma\alpha}_{\lambda\rho\beta}$ is conformally invariant.

Consideration of the expressions $\nabla[{}^{\lambda}{}_{\alpha}\nabla^{\mu}{}_{\beta}\nabla^{\nu}{}_{\gamma}]\phi$ and $\nabla[{}^{\lambda}{}_{\alpha}\nabla^{\mu}{}_{\beta}\nabla^{\nu}{}_{\gamma}]Z^{\alpha}$ leads to two further identities satisfied by the curvature twistor; these are:

$$(\delta^{\nu}{}_{\tau}K^{\lambda\mu\kappa}_{\alpha\beta\gamma} + \delta^{\lambda}{}_{\tau}K^{\mu\nu\kappa}_{\beta\gamma\alpha} + \delta^{\mu}{}_{\tau}K^{\nu\lambda\kappa}_{\gamma\alpha\beta})\nabla^{\tau}{}_{\kappa}\phi = 0$$

and

$$\nabla[{}^{\lambda}{}_{\alpha}K^{\mu\nu}_{\beta\gamma}]^{\kappa}{}_{\tau} + T^{\rho}{}_{\sigma}[{}^{\lambda\mu}_{\alpha\beta}K^{\nu}{}_{\gamma}]^{\sigma\kappa}_{\rho\tau} = 0$$

The corresponding tensor indenities are:

$$R_{[abc]}{}^{d} + \nabla_{[a}T_{bc]}{}^{d} + T_{[ab}{}^{x}T_{c]x}{}^{d} = 0$$

and

$$\nabla_{[a}R_{bc]d}{}^{x} + T_{[ab}{}^{y}R_{c]yd}{}^{x} = 0$$

where ∇_a is now a covariant derivative with torsion; note that in the first local twistor identity the $\nabla^{\tau}{}_{\kappa}\phi$ cannot be cancelled out since it does not span the space of two-index local twistors.

A torsion-free derivative $\diamondsuit^{\sigma}{}_{\rho}$ can be obtained from $\nabla^{\sigma}{}_{\rho}$; the defining equations are as follows:

$$\left.\begin{aligned} \diamondsuit^{\sigma}{}_{\rho}Z^{\alpha} &= \nabla^{\sigma}{}_{\rho}Z^{\alpha} - \frac{i}{2}\delta^{\alpha}{}_{\rho}Z^{\sigma} \\ \diamondsuit^{\sigma}{}_{\rho}W_{\beta} &= \nabla^{\sigma}{}_{\rho}W_{\beta} + \frac{i}{2}\delta^{\sigma}{}_{\beta}W_{\rho} \end{aligned}\right\} \tag{4.16}$$

and

$$\diamondsuit^{\sigma}{}_{\rho}\phi = \nabla^{\sigma}{}_{\rho}\phi \tag{4.17}$$

A simple calculation shows that: $[\diamondsuit^{\mu}{}_{\lambda}, \diamondsuit^{\sigma}{}_{\rho}]\phi = 0$. It also ensues that:

$$[\diamondsuit^{\mu}{}_{\lambda}, \diamondsuit^{\sigma}{}_{\rho}]Z^{\alpha} = -i\mathscr{K}^{\mu\sigma\alpha}_{\lambda\rho\beta}Z^{\beta} \tag{4.18}$$

where

$$\mathscr{K}^{\mu\sigma\alpha}_{\lambda\rho\beta} = K^{\mu\sigma\alpha}_{\lambda\rho\beta} - \tfrac{1}{4}T^{\mu\sigma\alpha}_{\lambda\rho\beta} \tag{4.19}$$

$\mathscr{K}^{\mu\sigma\alpha}_{\lambda\rho\beta}$ is the curvature twistor associated with the operator $\diamondsuit^{\sigma}{}_{\rho}$; evidently, $\mathscr{K}^{\mu\sigma\alpha}_{\lambda\rho\beta}$ does not vanish in general in flat space-time (since $T^{\mu\sigma\alpha}_{\lambda\rho\beta}$ is non-zero), and therefore $\diamondsuit^{\sigma}{}_{\rho}$ is non-integrable in flat space-time.

Appendix

Generation of Conformally Invariant Spinors

In this appendix, it will be shown that conformally invariant spinor expressions can be generated by use of the local twistor covariant derivative. It will be seen that some of the spinor expressions obtained here can be generalised; the main spinor results are stated in Lemmas AI, AII, and AIII.

A general (non-zero) local twistor defines at least one non-zero conformally invariant spinor, which is obtained by expressing the local twistor in terms of its spinor parts, and locating the spinor expression with the greatest number of 'contravariant' indices; consider, for example, a non-zero local twistor $W^{\alpha}{}_{\beta}{}^{\gamma}$, which may be expressed in terms of its spinor parts thus:

$$W^{\alpha}{}_{\beta}{}^{\gamma} = e^{\alpha}{}_{A}\, e_{\beta B'}e^{\gamma}{}_{C}W^{AB'C} + e^{\alpha A'}e_{\beta B'}e^{\gamma}{}_{C}W_{A'}{}^{B'C} + e^{\alpha}{}_{A}\, e^{B}{}_{\beta}e^{\gamma}{}_{C}W^{A}{}_{B}{}^{C} + \cdots$$

If $W^{AB'C}$ is non-zero, then it is conformally invariant; if $W^{AB'C} = 0$ then each of $W_{A'}{}^{B'C}$, $W^{A}{}_{B}{}^{C}$, $W^{AB'}{}_{C'}$ is conformally invariant, and if they are all zero the process may be repeated until a conformally invariant spinor is obtained (there must be at least one if $W^{\alpha}{}_{\beta}{}^{\gamma} \neq 0$). Applying these considerations to the twistor $K^{\mu\sigma\alpha}_{\lambda\rho\beta}$ shows that $\epsilon_{M'S'}\Psi_{B}{}^{A}{}_{LR}$ and $\epsilon_{RL}\bar{\Psi}^{B'}{}_{A'M'S'}$ are both conformally invariant. Forming the covariant derivative of $K^{\mu\sigma\alpha}_{\lambda\rho\beta}$ does not immediately lead to a new conformally invariant spinor due to the torsion terms, but these terms may be eliminated by suitable symmetry and skew-symmetry operations. When this procedure is carried out, a local twistor with one spinor part is obtained, and it can be shown that this spinor expression is essentially the spinor defined by the Bach tensor†. Hence, the spinor is termed the Bach spinor, and the twistor produced from it is the Bach twistor. The calculation is outlined below:

$$\nabla^{\nu}{}_{\tau} K^{\mu\sigma\alpha}_{\lambda\rho\beta} = W^{\nu\mu\sigma\alpha}_{\tau\lambda\rho\beta} + \text{torsion terms}$$

where:

$$\begin{aligned}
W^{\nu\mu\sigma\alpha}_{\tau\lambda\rho\beta} = {} & \\
& e^{T}{}_{\tau} e^{\nu N'} e^{L}{}_{\lambda} e^{\mu M'} e^{R}{}_{\rho} e^{\sigma S'} \{ e^{B}{}_{\beta} e^{\alpha A'} [\epsilon_{M'S'} (\epsilon_{A'N'} \Lambda \Psi_{BTLR} - \Phi_{TAN'A'} \Psi_{B}{}^{A}{}_{LR} \\
& + \nabla_{TN'} \nabla^{X}{}_{A'} \Psi_{XRLB}) + \epsilon_{LR} (\epsilon_{BT} \Lambda \bar{\Psi}_{A'M'N'S'} - \Psi_{TBN'B'} \bar{\Psi}^{B'}{}_{A'M'S'} \\
& + \nabla_{TN'} \nabla^{X'}{}_{B} \bar{\Psi}_{X'M'S'A'})] \\
& + e_{\beta B'} e^{\alpha}{}_{A} [-i \epsilon_{LR} \nabla^{B'}{}_{T} \bar{\Psi}_{A'M'N'S'} - i \epsilon_{N'}{}^{B'} \epsilon_{M'S'} \nabla^{X}{}_{A'} \Psi_{XRLT}] \\
& + e^{B}{}_{\beta} e^{\alpha}{}_{A} [i \epsilon_{M'S'} \nabla^{A}{}_{N'} \Psi_{BTLR} + i \epsilon_{T}{}^{A} \epsilon_{LR} \nabla^{X'}{}_{B} \bar{\Psi}_{X'M'N'S'}] \\
& + e_{\beta B'} e^{\alpha}{}_{A} [\epsilon_{N'}{}^{B'} \epsilon_{M'S'} \Psi_{T}{}^{A}{}_{LR} + \epsilon_{T}{}^{A} \epsilon_{LR} \bar{\Psi}^{B'}{}_{M'N'S'}] \}
\end{aligned} \tag{A.1}$$

Then

$$W^{[\nu\mu]\sigma\alpha}_{[\tau\lambda]\rho\beta} = B^{\nu\mu\sigma\alpha}_{\tau\lambda\rho\beta} = e^{T}{}_{\tau} e^{\nu N'} e^{L}{}_{\lambda} e^{\mu M'} e^{R}{}_{\rho} e^{\sigma S'} \{ {}^{B}{}_{\beta} e^{\alpha A'} \{ \epsilon_{M'N'} \epsilon_{TL} B_{BRA'S'} \}$$

where

$$B^{\nu\mu\sigma\alpha}_{\tau\lambda\rho\beta} = B^{[\nu\mu](\sigma\alpha)}_{[\tau\lambda](\rho\beta)} = B\binom{\nu\mu}{\tau\lambda}\binom{\sigma\alpha}{\rho\beta} \tag{A.2}$$

is the Bach twistor, and

$$B_{BRA'S'} = B_{(BR)(A'S')} = (\nabla^{X}{}_{A'} \nabla^{Y}{}_{S'} - P^{XY}_{A'S'}) \Psi_{BRXY}$$

is the Bach spinor.

† The Bach tensor is defined by:

$$B_{ab} = \nabla^{c} \nabla^{d} C_{cabd} - \tfrac{1}{2} P^{cd} C_{cabd} \qquad \text{(Szekeres, 1968)}$$

It is of interest in the present context since it transforms under conformal transformation as a conformal density of weight -2, i.e. $\hat{B}_{ab} = \Omega^{-2} B_{ab}$, and is algebraically independent of the Weyl tensor. It is represented in spinors by:

$$B_{ABA'B'} = Q_{ABA'B'} + \bar{Q}_{ABA'B'}$$

where

$$Q_{ABA'B'} = \bar{Q}_{ABA'B'} = (\nabla^{C}{}_{A'} \nabla^{D}{}_{B'} - P^{CD}_{A'B'}) \Psi_{ABCD}$$

The Bach tensor has the additional properties (which are readily seen in its spinor form), that: $B_{ab} = B_{(ab)}$, and $\nabla^{a} B_{ab} = 0$.

The spinor result can be generalised as follows:

Lemma AI

$(\nabla^A{}_{A'}\nabla^B{}_{B'} - P^{AB}_{A'B'})\chi_{AB\ldots L}$ transforms under a conformal rescaling as a conformal density of weight -2, where $\chi_{AB\ldots L}$ is a totally symmetric spinor with q indices ($q \geqslant 2$), and which is conformally invariant, i.e.

$$\chi_{AB\ldots L} = \chi_{(AB\ldots L)} = \hat{\chi}_{AB\ldots L}$$

The result can be proved directly by spinor methods, or by constructing the local twistor

$$Y^{\mu\sigma\alpha}_{\beta\gamma\delta\ldots\lambda} = Y^{[\mu\sigma]\alpha}_{(\beta\gamma\delta\ldots\lambda)} = Y^{[\mu\sigma]\alpha}_{[\beta\gamma](\delta\ldots\lambda)}$$

such that:

$$Y^{\mu\sigma\alpha}_{\beta\gamma\delta\ldots\lambda} = e^{\mu M'} e^{\sigma S'} e^B{}_\beta e^C{}_\gamma \ldots e^L{}_\lambda \{e^\alpha{}_A i \epsilon_{M'S'} \chi^A{}_{BC\ldots L} + e^{\alpha A'} \epsilon_{M'S'} \nabla^X{}_{A'} \chi_{XBC\ldots L}\}$$

and considering the derivative $\nabla^{(\nu}_{[\tau} Y^{\mu)}_{\beta]}{}^{\sigma\alpha}_{\gamma\delta\ldots\lambda}$.

Another example of a conformally invariant spinor expression is obtained by contracting the curvature twistor $K^{\mu\sigma\alpha}_{\lambda\rho\beta}$ with the totally skew twistor† $\epsilon_{\mu\sigma\alpha\delta}$; a local twistor with one spinor part only is then formed:

$$\epsilon_{\mu\sigma\alpha\delta} K^{\mu\sigma\alpha}_{\lambda\rho\beta} = 2i\Psi_{\lambda\rho\beta\delta}$$

where

$$\Psi_{\lambda\rho\beta\delta} = e^L{}_\lambda e^R{}_\rho e^B{}_\beta e^D{}_\delta \Psi_{LRBD} = \Psi_{(\lambda\rho\beta\delta)}$$

From this local twistor a hermitian, eight-index, valence $\left[\begin{smallmatrix}4\\4\end{smallmatrix}\right]$ local twistor can be defined:

$$\Psi_{\alpha\beta\gamma\delta} \bar{\Psi}^{\kappa\lambda\mu\nu} = \Psi^{\kappa\lambda\mu\nu}_{\alpha\beta\gamma\delta} = \bar{\Psi}^{\kappa\lambda\mu\nu}_{\alpha\beta\gamma\delta}$$

On forming the derivative: $\nabla^\sigma{}_\rho \Psi^{\kappa\lambda\mu\nu}_{\alpha\beta\gamma\delta}$ and taking skew-symmetries on the indices ρ, α and σ, κ the following spinor result is obtained:

Lemma AII

$\bar{\Psi}^{X'}{}_{A'B'C'} \nabla_{XX'} \Psi^X{}_{ABC} - \Psi^X{}_{ABC} \nabla_{XX} \bar{\Psi}^{X'}{}_{A'B'C'}$ is a conformal density of weight -2. This expression may be further refined to give the more general result:

Lemma AIII

The expression $\beta_{X'L'XX_{N'}} \nabla^{XX'} \alpha_{XA\ldots C} - \alpha_{XA\ldots C} \nabla^{XX'} \beta_{X'L'\ldots N'}$ is a conformal density of weight $(2n - 2)$, where $\beta_{X'L'\ldots N'}$ is a totally symmetric

† $\epsilon_{\mu\sigma\alpha\delta} = \epsilon_{[\mu\sigma\alpha\delta]}$; $\epsilon_{\mu\sigma\alpha\delta}\epsilon^{\mu\sigma\alpha\delta} = 24$. The spinor parts of this twistor are of the form $\epsilon_{MS}\, \epsilon^{A'D'}$.

spinor of valence $[^0_q]$, $q \geqslant 1$, and is a conformal density of weight n, and $\alpha_{XA\ldots C}$ is a totally symmetric spinor of valence $[^0_p]$, $p \geqslant 1$, and is also a conformal density of weight n.

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